IMAGING WITH HIGHER ORDER DIFFRACTION TOMOGRAPHY

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ABSTRACT

This paper describes several higher order models for the fields scattered by an object and the computational techniques involved. The Born and the Rytov approximations are easily extended to include higher order terms but like most series their range of convergence is limited. Another technique to be described is based on algebraic techniques and can be shown to always converge. Unfortunately, numerical instabilities limit this approach to objects with a refractive index change of less than 20-30%. This paper will describe numerical approaches for each of these higher order models It reports the region of convergence for the Born and Rytov series and objects that lead to stable solutions for the algebraic approach.

1. INTRODUCTION

To date the success of ultrasound (and microwaves) for quantitative medical imaging has been severly limited by the approximations needed to derive a reconstruction procedure. Two techniques have been used to reconstruct cross sectional images of an object from the scattered ultrasonic fields. B-scan imaging has certainly had the greater clinical success and generates an image of the object's reflectivity distribution by assuming that the refractive index is constant. As the refractive index (and thus the object's reflectivity) increases there is first geometric distortion and then severe artifacts due to multiple scattering.

Diffraction tomography provides a more quantitative approach to ultrasound imaging. It was first developed by Wolf [20] and then applied to ultrasound by Mueller [14].

First order diffraction tomography is based on an approximate solution to the wave equation. For the purposes of this paper we assume a wave equation of the form

$$(\nabla^2+k_0^2)u(\vec{r}) = -o(\vec{r})u(\vec{r}) \tag{1}$$

where k_0 is the average wavenumber of the media and is related to the average wavelength, λ, of the field by $k_0=2\pi/\lambda$. The complex amplitude of the field at position $\vec{r}=(x,y)$ is denoted by $u(\vec{r})$ and finally the object function is written $o(\vec{r})=k_0^2[n^2(\vec{r})-1]$ where $n(\vec{r})$ is the (complex) refractive index change of the object [10,12].

The differential equation is easily converted to the following integral equation

$$u_s(\vec{r}) = \int[u_s(\vec{r}')+u_0(\vec{r}')]o(\vec{r}')g(\vec{r}-\vec{r}')d\vec{r}' \tag{2}$$

where $u_s(\vec{r})$ is called the scattered field, $u_0(\vec{r})$ is the incident field and satisfies the equation

$$(\nabla^2+k_0^2)u_0(\vec{r}) = 0. \tag{3}$$

Finally $g(\vec{r}-\vec{r}')$ is the Green's function. It represents the solution of the wave equation for a point disturbance or the equation [12]

$$(\nabla^2+k_0^2)g(\vec{r}) = -\delta(\vec{r}). \tag{4}$$

The conventional approach to solve for the unknown object is to use one of two approximations in equation (2) so that the integral equation can be inverted. These two approximations are known as the Born and the Rytov approximations and both assume that the incident field is not significantly altered inside the object.

The Born approximation is realized by assuming that $u_0+u_s\approx u_0$ inside the integral. An approximate expression for the scattered field is

$$u_s(\vec{r}) \approx \int u_0(\vec{r}')o(\vec{r}')g(\vec{r}-\vec{r}')d\vec{r}' \tag{5}$$

and can be readily solved using the Fourier Slice Theorem as described in [10, 17, 4, 14, 11].

The Rytov approximation, on the other hand, is found by writing the field as

$$u(\vec{r}) = e^{\phi(\vec{r})} = e^{\phi_0(\vec{r})+\phi_s(\vec{r})} \tag{6}$$

where $u_0(\vec{r})=e^{\phi_0(\vec{r})}$. The scattered field is then expressed as

$$u_0(\vec{r})\phi_s(\vec{r}) = \int g(\vec{r}-\vec{r}')u_0(\vec{r}')[(\nabla\phi_s(\vec{r}'))^2+o(\vec{r}')]d\vec{r}'. \tag{7}$$

The Rytov approximation is then made by assuming that the term in brackets above can be approximated by

$$(\nabla\phi_s(\vec{r}'))^2+o(\vec{r}') \approx o(\vec{r}'). \tag{8}$$

The scattered field is now written

$$u_0(\vec{r})\phi_s(\vec{r}) \approx \int g(\vec{r}-\vec{r}')u_0(\vec{r}')o(\vec{r}')d\vec{r}' \tag{9}$$

and using the Fourier Slice Theorem this equation can be easily inverted to find an estimate for the object.

The Born and the Rytov approximation assume that the scattered field, u_s, is small compared to the

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incident, u_0. These approximations severely limit the types of objects that can be successfully imaged. As shown in [16, 2] only objects with a product of radius (in wavelengths) and the change in refractive index, n−1, less then 0.2 can be successfully imaged with the Born. While the Rytov approximation limits good reconstructions to objects with a refractive index change of less than a few percent.

The remainder of this paper will investigate two different iterative techniques to better model the scattered fields. They can be briefly described as a fixed point method, where each estimate is assumed better then the previous and is used to calculate an even better estimate, and an algebraic approach, where the wave equation is written as a matrix equation and linear algebra techniques are used to solve it. With a better understanding of these techniques it is then possible to derive better reconstruction techniques.

Two other approaches to generate better reconstructions will not be presented here. Ray tracing assumes geometric optics holds and solves a sparse matrix equation. As reported in [1] this technique has problems with refractive indices greater than 10%. In addition a second technique based on Generalized Radon Transforms has been presented by Beylkin [3] and will not be discussed here.

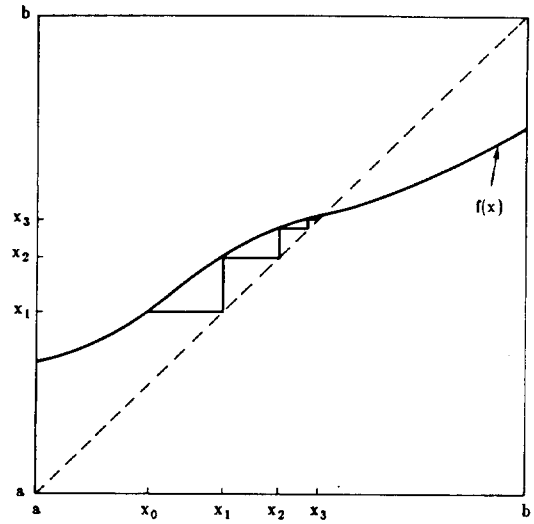

Figure 1. From an initial guess, x_0, a better estimate for the solution of $x=f(x)$ is found by iterating $x_{i+1}=f(x_i)$.

2. FIXED POINT METHODS

One way to solve an equation of the form

$$x = f(x) \tag{10}$$

is to pick a value, x_0, and to assume that

$$x_1 = f(x_0) \tag{11}$$

is closer to the correct answer then the original guess. This iterative procedure

$$x_{n+1} = f(x_n) \tag{12}$$

can then be repeated until the answer is as exact as necessary. This procedure is diagramed in Figure 1.

It can be shown that the procedure will always converge is there exists a region $a<x<b$ where

$$|f'(x)| < 1 \tag{13}$$

and an initial guess, x_0, can be made that falls within this region [17, 18]. If this condition can not be met then the iteration series will probably diverge.

The technique can be easily applied to the wave equation, (2). For the Born approximation the first iteration is given by

$$u_1(\vec{r}) = \int u_0(\vec{r}')o(\vec{r}')g(\vec{r}-\vec{r}')d\vec{r}' \tag{14}$$

and higher order terms are written

$$u_{i+1}(\vec{r}) = \int [u_0(\vec{r}')+u_i(\vec{r}')]o(\vec{r}')g(\vec{r}-\vec{r}')d\vec{r}' \tag{15}$$

Rewriting this equation so that

$$u^{i+1}(\vec{r}) = \int u^i(\vec{r}')o(\vec{r}')g(\vec{r}-\vec{r}')d\vec{r}' \tag{16}$$

and

$$u_i(\vec{r}) = \sum_{j=1}^{i} u^j(\vec{r}) \tag{17}$$

it is easy to see that the fixed point technique accurately models what is known as the multiple scattering approach.

The same technique also applies to the Rytov approximation. The Rytov series can be written as [6]

$$\phi_{s,i+1} = \frac{1}{u_0(\vec{r})}\int g(\vec{r}-\vec{r}')u_0(\vec{r}')[(\nabla\phi_{s,i}(\vec{r}'))^2+o(\vec{r}')]d\vec{r}' \tag{18}$$

where $\phi_{s,0} = 0$.

To implement these fixed point techniques on a digital computer a number of "tricks" are needed to quickly compute each iteration. Each of these techniques will be briefly mentioned and the reader is referred to [17] for more of the implementation details.

Certainly the most significant implementation detail is to realize that the integrals in (15) and (18) represent a convolution. Because of this each iteration is best computed in the frequency domain. This is relatively straightforward as long as each matrix is adequately zero padded so that convolutions are aperiodic instead of circular. Even with the appropriate zero padding it is possible to reduce the number of operations to compute a 128x128 matrix by a factor of 600. With a careful implementation we were able to compute each iteration of a 128x128 matrix in one to two seconds on a Floating Point Systems AP120B with 64K words of memory.

A second problem is presented by a singularity in the Green's function at the origin. This turns out not to be a problem if the function that is convolved with the Green's function is smooth within each pixel. With this assumption it is possible to compute the average value of the pixel when the Green's function is convolved with a smooth function.

Finally the derivative required as part of the Rytov series requires careful attention. The obvious way to compute the ∇^2 operation is to perform it in the frequency domain. Unfortunately this technique does not work for this application because of the zero padding needed for the FFT convolution. Adding zeroes to the data creates discontinuities at the edge of the array that lead to large errors. In our experience a better technique is to calculate a four point (two in each direction) derivative at each point. While this is less accurate from a signal processing point of view it does generate less artifacts.

Two questions are most important when discussing an iterative technique. First does it converge to the correct answer and secondly under what conditions does this happen? The first question is easily answered by comparing the iterative answer to the exact field as calculated by solving the boundary conditions [12]. The convergence of the series for one object is demonstrated in Figure 2.

Describing conditions for convergence is more difficult. For a one dimensional Taylor series the region of convergence can usually be described as a region around the origin. An object used to study the convergence of the Born or Rytov series is decidedly more complex and can require an infinite number of parameters to describe it. For this study we limited our objects to those that are a function of two parameters, in this case the radius and refractive index of a cylinder.

For each object the decision on convergence is made by calculating a number of terms and noting whether the magnitude of the change in field is increasing or decreasing. Empirically it was determined that it was only necessary to look for a monotonic sequence of four, in the Born case, and six, in the Rytov case, terms to establish convergence of divergence.

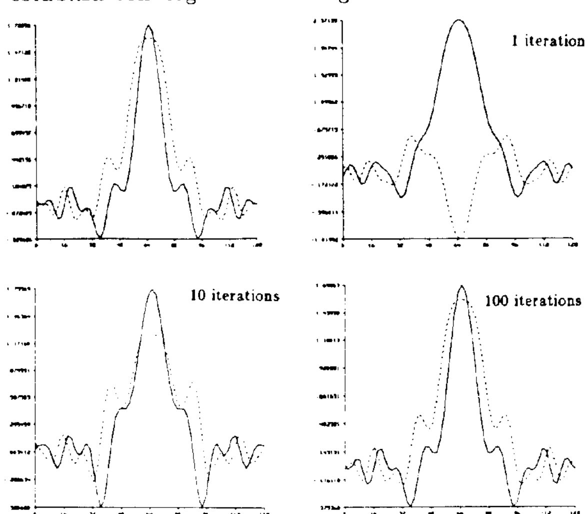

Figure 2. This figure shows (upper left) the exact scattered field from a cylinder of radius 2λ and refractive index 1.10. The other plots show the Born iterated field after 1, 10 and 100 terms. In each case the real part of the field is a solid line and the imaginary part is shown as a dashed line.

The results from this study of the Born and the Rytov convergence are shown in Figure 3. As might be expected from the behavior of the first order approximations both series converge for small objects with small refractive index changes but as the object's radius and refractive index increase each series starts to diverge.

The series convergence was also studied in the presense of attenuation. Figure 4 shows the region of convergence when the object is attenuating. As expected the region of convergence gets smaller as the attenuation increases.

On the other hand if the object and the media in which it is embedded are both attenuating then the average attenuation can be incorporated into the wavenumber, k_0. Now the Green's function represents the field scattered by a point disturbance in an attenuating media and the resulting convergence is shown in Figure 5. The region of convergence gets larger now because the attenuating media reduces the effect of multiple scattering. In both cases the results using the Rytov approximation are similar.

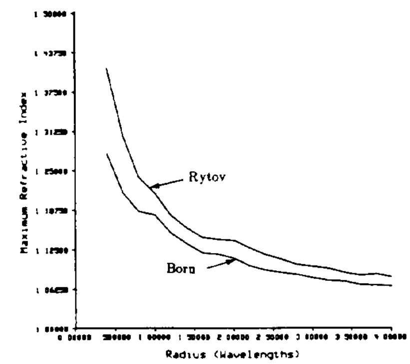

Figure 3. The region of convergence of the Born and Rytov series is compared here.

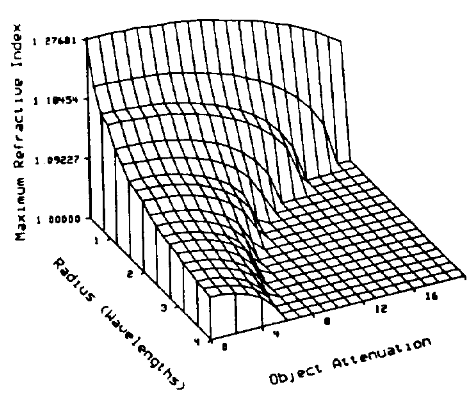

Figure 4. The region of convergence for the Born series is shown here in the presense of an attenuating object. Attenuation is plotted in terms of nepers (1 neper represents the attenuation of the field by 63% per wavelength).

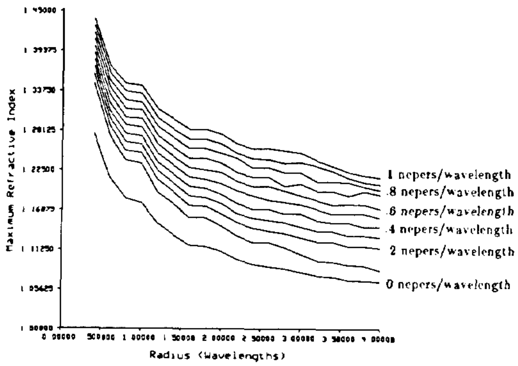

Figure 5. The region of convergence for the Born series is shown here when the average attenuation of the object and the surrounding media is included in the model.

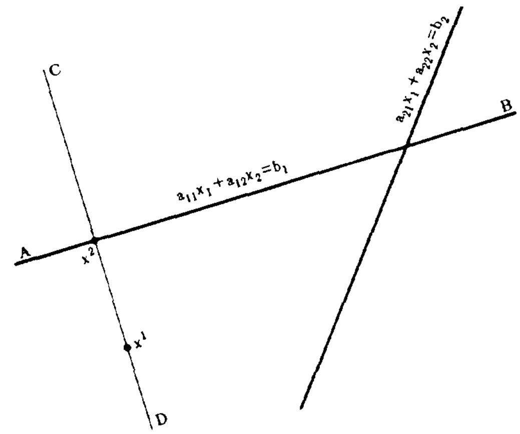

Figure 6. By projecting the initial solution x^1 onto the AB hyperplane a better solution can be found.

3. AN ALGEBRAIC APPROACH

A second approach to higher order diffraction tomography was first proposed by Johnson [7, 19, 8] and is based on linear algebra. By appropriately sampling the original integral equation it is possible to write a matrix equation. A solution to the matrix equation is then found by considering each row of the matrix to represent the equation for a hyperplane and then iteratively projecting an initial guess onto each hyperplane.

In the algebraic approach the object and the field are sampled on an NxN square grid and then assigned as elements of a one dimensional N^2xN^2 element vector. Doing this the integral equations of (2) can be rewritten as

$$\mathbf{u}_s = \mathbf{A}\mathbf{u}_0 + \mathbf{A}\mathbf{u}_s \qquad (19)$$

where the **A** matrix has N^2 elements on a side and incorporates the Green's function and the object. The equation can then be formulated as a standard matrix equation or

$$(\mathbf{A}-\mathbf{I})\mathbf{u}_s = \mathbf{A}\mathbf{u}_0. \qquad (20)$$

Here both the **A** matrix and the vector $\mathbf{u}_0$ are known so we will consider the more general case of $\mathbf{A}\mathbf{x}=\mathbf{b}$.

At this point it is important to realize that it is not necessary to calculate the inverse of **A**; but only one particular **x** that solves the equation. This distinction is important because for a small (by medical standards) 64x64 image the **A** matrix has over 16 million non-zero entries.

A very successful solution to this problem [5] is to consider the solution to be at the intersection of all the hyperplanes. An initial guess is then refined by finding the nearest point on one of the hyperplanes. This is done using the formula

$$x^{j+1} = x^j - \frac{<a_i,x^j> - b_i}{<a_i,a_i>} a_i \qquad (21)$$

where a_i is the i^{th} row of the **A** matrix, b_i is the i^{th} element and $<,>$ represents a dot product. This process is diagramed in Figure 6 and can be seen to never diverge because the result of each projection, x^2, is always closer to the intersection point than any other point on the line CD.

Certainly the biggest advantage of the Kaczmarz approach to solving the matrix equation is that only one row of the **A** must be present at a time. Since each row of the **A** matrix is relatively inexpensive to compute it is not necessary to store the entire matrix. Doing this we were able to implement a routine to project onto all of 64x64 hyperplanes in one CPU second on a Cyber 205 super-computer (real cost 21 cents).

The theory predicts that this method will never diverge and this is confirmed in numerical simulation. Unfortunately there is no guarantee that the method will converge in a reasonable rate and this leads to problems. The plots of Figure 7 shows the scattered fields after 32 iterations. While the exact limits of the Kaczmarz approach are difficult to define it is possible to say that the Kaczmarz algorithm seems to converge for refractive indices up to 1.4 when a sampling interval of $.1\lambda$ is used and 1.2 with a sampling interval of $.25\lambda$.

4. CONCLUSIONS

In this paper we have presented two different techniques for modeling the fields scattered by an object. The fixed point method can be used to better the Born and Rytov approximations but like the first order approximations the series have a small region of applicability. A second approach based on linear algebra techniques was also studied but its convergence properties are difficult to define.

Both approaches have been used to develop reconstruction schemes. Workers in high energy physics [9, 13, 15] have presented iterative techniques but they are all based on the Born series. This has serious

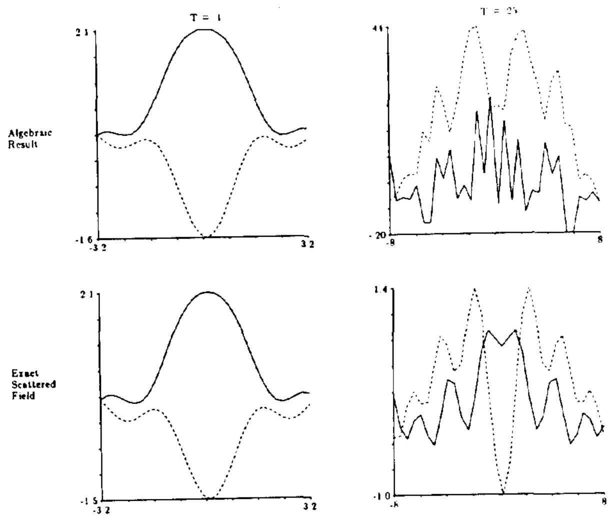

Figure 7. The field calculated by the algebraic approach (top) is compared to the exact field (bottom). A cylinder with a refractive index of 1.3 and a radius of .4λ (left) vs. 1λ (right) is modeled here.

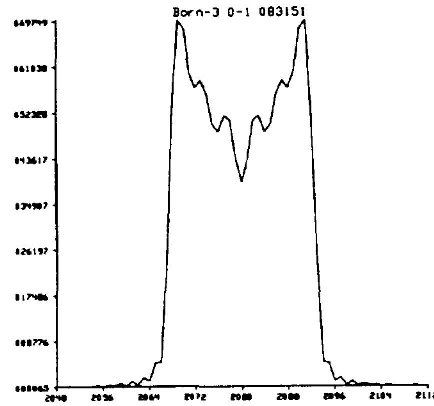

Figure 8. An ideal first order reconstruction is shown for an object with radius 3λ and refractive index 1.083. The Born series will converge for this object but if either the refractive index or the radius is increased then the series will diverge.

ramifications because the region of convergence of the Born series is not much greater than where the first Born approximation produces good results. This is shown in Figure 8 where a first order Born reconstruction is shown for an object that is on the boundary of convergence for the Born series. Thus if a first order reconstruction is worse then this one then an iterative technique based on the Born series will not converge. Up to the limit imposed by the Born series an iterative technique should be more accurate.

Reconstruction techniques based on the algebraic approach are also difficult because both the field inside the object and the object itself are unknown. Thus the matrix equation becomes non-linear and the computations are much more difficult [8].

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